CORRESPONDENCE.

ON THE VALUE OF CONTINGENT REVERSIONARY INTERESTS.

To the Editor of the Assurance Magazine.

SIR,—In one of the first papers read before the Institute of Actuaries, Mr Jellicoe pointed out "the method of procedure which should be observed in determining the value of life contingencies, where the risk was of an isolated character, as distinguished from cases in which the usual considerations of average presented themselves." The paper was not, I believe, printed for the use of the members, but a short account of it is given in the *Post Magazine* of the 10th March, 1849. This was followed by one from Mr. Hardy, "On the values of annuities which are to pay certain given rates of interest on the purchase money during the whole term of their continuance, and to replace their original values, on their expiration, at certain other given rates"—where for the first time also is given, in an intelligible form, an expression for the value of an annuity certain under such conditions; in accordance with this a set of tables was constructed, and appended to the paper read before the Institute 25th November, 1850.

In Vol. II. page 159 of the *Assurance Magazine* will be found a further communication from Mr. Jellicoe on contingent reversions, where, after alluding to the labours of Mr. Sang on the same subject, he speaks of marketable securities *as constituting investments to be made not subject to any contingency whatever, but as securing to the holder of them, in any case, a given rate of interest so long as he retains them, and reproducing the capital invested when such interest shall cease to be paid or to accrue.* In this point of view it is clear that the price a purchaser should pay for an annuity or reversion would be such as to enable him to get rid of the contingencies, if he wished to do so, without losing any of the usual advantages derivable from the purchase.

It may appear, and probably is, unnecessary to attempt to add anything to what has been so well done on this interesting subject; but, considering the importance of it to actuaries of the present day, owing to the frequency with which applications for loans on reversionary interests are now made to Insurance Offices, I have thought it may be useful once more to advert to it.

I propose to work out the two cases of contingent reversions given by Mr. Jellicoe, and to append a short table to each showing the marketable value of such reversions—that is, securing at the expiration of the contingency the capital advanced, and providing in the interval for the interest stipulated for.

The formula given by Mr. Griffith Davies for determining the value of an annuity on an isolated life is $\frac{1}{d+p}-1$. That given by Mr. Jellicoe as the present value of £1 annuity payable during the life of A, to commence at the death of B, is $\frac{1}{d+p}-(1+\text{AB})$, the truth of which is sufficiently evident without demonstration.

The annexed table, constructed from this formula, is made up of Nos. 3 and 5 of Davies's *Tables of Life Contingencies*—No. 3 giving $\frac{1}{d+p}-1$

at 5 and 6 per cent. with the Northampton premium, as charged by the Equitable; No. 5, the Equitable experience at $3\frac{1}{2}$ per cent. The former may be adapted to any other rates of premium, by selecting the nearest age in the Northampton Table corresponding with the premium proposed to be charged; the latter is taken to represent the fair value of an annuity on the joint existence of two lives, and which is intended to form part of the sum advanced.

If $s=\frac{1}{d+p}-1-\text{AB}$, then $s+\text{AB}=\frac{1}{d+p}-1$, and $s+\text{AB}+a=$ the sum to be assured; $p(s+\text{AB}+a)=$ the annual premium for ditto; $s+\text{AB}+p(s+\text{AB}+a)=$ the purchase-money, the price of an annuity during the joint lives, and the first year's premium, making together the sum advanced; $i\{s+\text{AB}+p(s+\text{AB}+a)\}=$ annual interest; and consequently $p(s+\text{AB}+a)+i\{s+\text{AB}+p(s+\text{AB}+a)\}=a$.

Suppose A, aged 25 years *next* birthday, to be entitled to an estate, producing £1,000 per annum, on the death of B, aged 65 years *last* birthday, and to require an immediate advance of £1,000, by way of deferred annuity. To determine the amount of such annuity, so as to allow the lender 5 per cent. interest besides the Northampton premium,

$$\frac{1}{d+p}-1 = 12{\cdot}958$$

$$\text{AB} \quad = \quad 8{\cdot}331 \quad \text{(Equitable } 3\tfrac{1}{2} \text{ per cent.)}$$

4·627 =Value of A's interest per £.

216123 =Reciprocal of ditto, or ratio of annuity
1000

216·123 =Deferred annuity to be granted.

216·123 × 8·331 = 1800·521 = Price of annuity during joint lives
133·8

1728984
64837
6484
216

1800·521

1000· =Sum paid to borrower
216·123=One year's annuity

3016·644=Sum to be insured.

3016·644 × ·02404 = 72·520=Annual premium
40420·

6033288
1206658
12066

72·52010

1000·
1800·521

2873·041=Sum advanced
05

143·65205=Interest on ditto
72·520 =Annual premium

216·172 =Annuity as above (very nearly).

TABLE I.—*Showing the value of an Annuity of £1, on A, after the death of B; allowing the purchaser a given rate of interest on the sum advanced, besides the premium necessary to secure his capital by a life assurance according to the Northampton 3 per Cent. Table.*

AGE of A.	AGE of B.	INTEREST, 5 per cent.		INTEREST, 6 per cent.	
		Value of £1 Annuity on A after B.	Annuity which £1 will purchase.	Value of £1 Annuity on A after B.	Annuity which £1 will purchase.
20	50	1·351	·7402		
	55	2·469	·4050	·817	1·224
	60	3·655	·2736	2·003	·4993
	65	4·798	·2009	3·326	·3007
	70	6·373	·1569	4·721	·2118
	75	7·816	·1279	6·164	·1622
	80	9·327	·1072	7·675	·1303
25	55	2·188	·4570	·632	1·5823
	60	3·336	·2998	1·780	·5618
	65	4·627	·2161	3·071	·3256
	70	5·993	·1669	4·437	·2254
	75	7·411	·1349	5·855	·1708
	80	8·903	·1123	7·347	·1361
30	60	2·980	·3356	1·528	·6545
	65	4·233	·2362	2·781	·3596
	70	5·570	·1795	4·119	·2428
	75	6·964	·1436	5·512	·1814
	80	8·436	·1185	6·984	·1432
35	65	3·771	·2652	2·431	·4114
	70	5·074	·1971	3·734	·2678
	75	6·443	·1552	5·103	·1960
	80	7·897	·1266	6·557	·1525
40	70	4·528	·2208	3·312	·3019
	75	5·846	·1711	4·630	·2160
	80	7·278	·1374	6·062	·1650
45	75	5·226	·1914	4·139	·2416
	80	6·619	·1511	5·533	·1807
50	80	5·909	·1692	4·960	·2016

It will be seen from the foregoing table that a difference of not less than thirty years is taken between the age of A, "the tenant in reversion," and that of B, "the tenant in possession." Where the difference is less than thirty years, the annuity would either have a negative value, or be so small as almost to preclude the possibility of a loan being carried through upon the conditions here stated.

The other formula given by Mr. Jellicoe in the paper alluded to is, $1-(p+d)(1+\text{AB})$, as the present value of £1 to be paid on the decease of B if A survive; p being here the premium of insurance on A against B.

If we have the value of £1 payable upon A dying before B, and £1 upon B dying before A, the two are together equal to the value of £1 payable on the failure of their joint lives. The expression for this, in Mr.

Milne's notation, is, $\mathfrak{AB}=\underset{1}{\mathfrak{AB}}+\underset{1}{\mathfrak{BA}}$ $\therefore$ $\underset{1}{\mathfrak{BA}}=\mathfrak{AB}-\underset{1}{\mathfrak{AB}}$; and since $\mathfrak{AB}=1-(1-v)(1+\mathrm{AB})=1-d(1+\mathrm{AB})$, and $\underset{1}{\mathfrak{AB}}$, the single premium for insuring £1 upon A dying before B, $=p(1+\mathrm{AB})$ $\therefore$ $\underset{1}{\mathfrak{BA}}=1-(p+d)(1+\mathrm{AB})$.

The formula $\mathfrak{AB}-\underset{1}{\mathfrak{AB}}$ is better adapted for calculation than $1-(p+d)(1+\mathrm{AB})$, because the value of $\mathfrak{AB}$ may be found for any given rate of interest from Orchard's *Assurance Premiums*, and $\underset{1}{\mathfrak{AB}}$ (where the Northampton or Carlisle 3 per Cent. premiums are used) can be obtained at once from Davies's *Tables*, and from those by Messrs. Gray, Smith, and Orchard. But then, by using the formula $\mathfrak{AB}-\underset{1}{\mathfrak{AB}}$, the condition cannot be fulfilled which should be stipulated for in transactions of this kind—that an annuity equal in amount to the reversionary annuity should be purchased during the joint lives of A and B, which can only be done by estimating the value of such annuity at the same rate of interest as is required for the loan.

We have seen that in the example given of the value of a reversionary annuity two rates are introduced, 5 per cent. being required for the advance of money, and $3\frac{1}{2}$ per cent. only allowed for the joint life annuity. But by using the expression $1-(p+d)(1+\mathrm{AB})$, any two rates may be introduced—AB may be taken, as above, at $3\frac{1}{2}$ per cent., and d at 5 per cent.

Taking the same ages as before, what sum should A pay at the death of B for an immediate advance of £1,000, allowing the lender 5 per cent. interest, besides the Northampton premium for insuring the life of A against that of B?

The value will be $\dfrac{1000}{1-(p+d)(1+\mathrm{AB})}$

$$
\begin{array}{rl}
p=\cdot01637 & \quad 1+\mathrm{AB}=9\cdot331 \text{ (Equitable } 3\tfrac{1}{2} \text{ per cent.)} \\
d=\cdot04762 & \\
\hline
\cdot06399 & \\
133\cdot9 & \\
\hline
\cdot57591 & \\
1920 & \\
192 & \\
6 & \\
\hline
\cdot59709 & \\
\hline
\cdot40291 & =1-(p+d)(1+\mathrm{AB})
\end{array}
$$

$\dfrac{1000}{\cdot4029}=2482$, the sum required.

$\dfrac{2482-1000}{9\cdot331}=158\cdot825$, the amount of annuity to be purchased during the joint lives.

Then proceeding as in the former example—

158·825 × 8·331	=1323·171=Price of joint life annuity
133·8	1000·000=Sum paid to borrower
	158·825=One year's annuity
1270600	
47647	2481·996=Sum to be insured.
4765	
159	
1323·171	

2482 × ·01637	= 40·62 =Annual premium
73610·	1000·
	1323·171
2482	
1489	2363·791=Sum advanced
74	·05
17	
	118·18955=Interest on ditto
40·62	40·62 =Annual premium
	158·809 =Annuity as above.

Taking AB at 5 per cent. and using the formula $\mathfrak{AB}-\mathfrak{A}\underset{1}{\mathfrak{B}}$, we find, from Orchard's *Tables* (AB being=7·528), $\mathfrak{AB}$=·59391; and, by Davies's *Tables*, Northampton 3 per Cent., we have $\mathfrak{A}\underset{1}{\mathfrak{B}}$=·13701 ∴ $\mathfrak{AB}-\mathfrak{A}\underset{1}{\mathfrak{B}}$ =·4569; and, if the sum advanced be 1,000, the equivalent sum to be paid on the death of B if A survive, will be $\frac{1000}{\cdot 4569}$=2188·663, which is also the sum to be assured on A against B, the single premium for which will be 2188·663 × ·13701=299·869. We have consequently 2188·663 payable at the death of the joint lives, the present value of which is,

2188·663 × ·59391=1299·869
Deduct premium of insurance= 299·869

Leaves £1000· the sum advanced.

The same method of treating all isolated cases of annuities and reversions, whether absolute or contingent, immediate or deferred, should be used whenever it is desired to arrive at the *market value* of such securities.

Thus in the case of a simple reversion the formula will be $1-d(1+\mathrm{A})$; and although values so obtained appear very small at the early ages of life, they will be found to approximate to the ordinary tabular values at the more advanced ages.

Taking the sum in reversion as £1, and using Davies's Equitable Experience, interest 5 per cent., it will be seen that

At age	50	the market	value is	·32005,	and the	ordinary	value	·40944
„	60	„	„	·45326	„	„	„	·51060
„	70	„	„	·60050	„	„	„	·63077
„	80	„	„	·75218	„	„	„	·76360

TABLE II.—*Showing the value of £1, payable on the death of B, if A survive; allowing the purchaser a given rate of interest on the sum advanced, besides the premium necessary to secure his capital by an insurance on A against B, according to the Northampton 3 per Cent. Table.*

Age of B.	Age of A.	Interest, 5 per cent.		Interest, 6 per cent.	
		Value of £1 on B against A.	* Amount per £ to be insured on A against B.	Value of £1 on B against A.	* Amount per £ to be insured on A against B.
80	50	·6348	1·575	·5912	1·691
	45	·6508	1·536	·6065	1·649
	40	·6628	1·509	·6181	1·618
	35	·6738	1·484	·6289	1·590
	30	·6807	1·469	·6356	1·573
	25	·6856	1·459	·6402	1·562
	20	·6903	1·449	·6446	1·551
75	45	·5430	1·842	·4862	2·057
	40	·5586	1·790	·5010	1·996
	35	·5728	1·746	·5148	1·942
	30	·5822	1·718	·5238	1·909
	25	·5884	1·700	·5296	1·888
	20	·5941	1·683	·5349	1·869
70	40	·4595	2·176	·3901	2·563
	35	·4752	2·104	·4050	2·469
	30	·4870	2·053	·4161	2·403
	25	·4946	2·022	·4231	2·363
	20	·5011	1·996	·4290	2·331
65	35	·3803	2·629	·2982	3·353
	30	·3939	2·539	·3111	3·214
	25	·4029	2·482	·3191	3·134
	20	·4100	2·439	·3254	3·073
60	30	·3048	3·281	·2106	4·748
	25	·3148	3·177	·2194	4·558
	20	·3226	3·100	·2260	4·425
55	25	·2348	4·259	·1291	7·746
	20	·2429	4·117	·1357	7·369
50	20	·1668	5·995	·0495	20·202

* These values are the reciprocals of the numbers in the previous columns, and are also the reversions of which £1 is the present value.

I am, Sir,

Yours truly,

ROBERT TUCKER.

Lombard Street, 21st October, 1854.